Einführung in die Quantenoptik I

Wintersemester 2014/15
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Problem 3.1 – Vacuum energy (7 points)

In the energy of the quantized radiation field, we often 'neglect' the contribution of the zero-point energy of each field mode, given by $\frac{1}{2}\hbar\omega_k$ per mode. This 'small term' leads, in the vacuum state of the electromagnetic field, to an energy density u_{vac} that depends on the cutoff momentum k_c as follows

$$u_{\text{vac}} = C\hbar c k_c^4 \tag{3.1}$$

where C is a numerical constant 'of order unity'. (o) Calculate the constant C by counting plane wave modes in a 'quantization volume' V (two polarization modes per wave vector) and using a spherical cutoff in k-space: $|\mathbf{k}| \le k_c$. (i) Fix the cutoff at the energy scale $E_{\text{GUT}} = \hbar c k_c$ for the 'unification' of the fundamental interactions (except gravity, look up on the Internet the keyword 'grand unified theory') and make an estimate for the corresponding vacuum energy. Express this number in proton masses (times c^2) per cubic meter. (ii) Look up the length scale $\ell_{\text{Planck}} = 1/k_c$ for the unification of gravity, relativity, and quantum theory and compare the corresponding vacuum energy to case (ii). (iii) Somebody told you that the density of 'dark energy' in the vacuum (responsible for the accelerated expansion of the Universe, Nobel prize in physics 2011) is of the order of one proton rest mass (times c^2) per cubic meter. Find out more details on this and compare to the energy density of the cosmic microwave background.

The vacuum energy density predicted by quantum electrodynamics (one of the simplest quantum field theories) is *very, very* different from that attributed to dark energy, and there is no good explanation yet for this.

Problem 3.2 – Photon wavepackets (7 points)

In the lecture, we often use plane wave modes to quantize the field. They lead to photon operators $a_{\mathbf{k}\sigma}$ with the commutation relations (Fourier integrals with wave vector $\mathbf{k}$ and polarization index σ):

$$\left[a_{\mathbf{k}\sigma},\, a^\dagger_{\mathbf{k}'\sigma'}\right] = \delta_{\sigma\sigma'}(2\pi)^3\delta(\mathbf{k}-\mathbf{k}') \tag{3.2}$$

In the following, we deal only with the 'positive-frequency part' $\mathbf{E}^{(+)}(\mathbf{r})$ of the field operator:

$$\mathbf{E}^{(+)}(\mathbf{r}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0}} a_{\mathbf{k}\sigma}\, \mathbf{e}_{\mathbf{k}\sigma}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \tag{3.3}$$

(i) Show that we can construct the mode operator from the overlap integral

$$a_{\mathbf{k}\sigma} = \frac{1}{\mathcal{N}_k} \int \mathrm{d}V\, \mathbf{f}^*_{\mathbf{k}\sigma}(\mathbf{r}) \cdot \mathbf{E}^{(+)}(\mathbf{r}) \tag{3.4}$$

where $\mathbf{f}^*_{\mathbf{k}\sigma}(\mathbf{x})$ is the classical plane-wave mode and $\mathcal{N}_k$ a normalization factor that you are invited to calculate.

(ii) We know that plane waves are no proper field modes (because they are infinitely extended), but wave packets are. Consider two normalizable mode functions $\mathbf{f}$ and $\mathbf{g}$ and define

$$a_{\mathbf{f}} = \int \mathrm{d}V\, \mathbf{f}^*(\mathbf{r}) \cdot \mathbf{E}^{(+)}(\mathbf{r}) \tag{3.5}$$

and similarly for $a_{\mathbf{g}}$. Try to find out under which conditions the following relation holds between the commutator and the standard overlap integral

$$\left[a_{\mathbf{f}},\, a^\dagger_{\mathbf{g}}\right] = \mathcal{N}_{\mathbf{f}}\mathcal{N}_{\mathbf{g}} \int \mathrm{d}V\, \mathbf{f}^*(\mathbf{r}) \cdot \mathbf{g}(\mathbf{r}) \tag{3.6}$$

where the $\mathcal{N}_{\mathbf{f}}$, $\mathcal{N}_{\mathbf{g}}$ are again normalization factors.

(iii) Choose $\mathbf{g} = \mathbf{f}$ and fix the normalization such that the commutator $\left[a_{\mathbf{f}},\, a^\dagger_{\mathbf{f}}\right] = 1$. Argue that the state $a^\dagger_{\mathbf{f}}|\mathrm{vac}\rangle$ is normalized and contains exactly one photon.

Hint. It is useful to work with the Fourier transformations of $\mathbf{f}$ and $\mathbf{g}$. The photon number operator in the plane-wave basis is given by

$$N = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \sum_\sigma a^\dagger_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma} \tag{3.7}$$

Problem 3.3 – Fock space (6 points)

Collect from the lecture, books about quantum optics, and the Web information about the Fock–Hilbert space of quantum electrodynamics and write a one-page essay about this concept. As an additional question, try to think how the thermal state of the radiation field can be defined in a proper way in this frame work (issues of convergence etc.).