Einführung in die Quantenoptik I

Wintersemester 2014/15
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Problem 4.1 – Thermal radiation, exactly (12 points + 5 bonus points)

In the lecture, you have seen creation and annihilation operators for photons with a definite frequency. In this problem, we consider one (or two) modes with an energy operator

$$H = \hbar\omega a^\dagger a \tag{4.1}$$

(1) Show that the density operator ρ is diagonal in the number state basis where

$$\rho = \frac{\exp[-H/(k_B T)]}{Z_1} \tag{4.2}$$

(2) Compute the partition function (*Zustandssumme*) Z_1 (index 1 for 'one-mode').

(3) Show that the mean energy $\langle H \rangle$ and the mean photon number $\bar{n} = \langle a^\dagger a \rangle$ are given by the Bose-Einstein distribution:

$$\langle H \rangle = \hbar\omega\,\bar{n}\,, \qquad \bar{n} = \frac{1}{\exp[\hbar\omega/(k_B T)] - 1} \tag{4.3}$$

(4) The fluctuation Δn of the photon number is another quantity to characterize the so-called 'photon statistics' (i.e., the probability p_n to measure n photons in the thermal state ρ. Compute a formula for p_n and show that

$$\Delta n = [\bar{n}(\bar{n}+1)]^{1/2} = \frac{1}{4\sinh^2[\hbar\omega/(2k_B T)]} \tag{4.4}$$

Discuss the limiting cases $\hbar\omega \ll k_B T$ ('quantum regime') and $\hbar\omega \gg k_B T$ ('classical regime') and try give an interpretation.

(4') Not to forget (and easy to show): vanishing mean values

$$\langle a \rangle = \langle a + a^\dagger \rangle = \langle a\,\mathrm{e}^{\mathrm{i}\theta} + a^\dagger \mathrm{e}^{-\mathrm{i}\theta} \rangle = 0 \tag{4.5}$$

(5) For two modes with operators a, b (frequencies ω_1 and ω_2), show that the partition function factorizes:

$$Z_2 = Z_1(\omega_1) Z_1(\omega_2) \tag{4.6}$$

(6) Conclude that for the full field, one can construct the (natural) logarithm of the partition function as an intensive quantity (i.e., having a finite limit as the volume $V \to \infty$):

$$\frac{1}{V} \log Z_\infty = - \int đ\kappa \, \log(1 - \mathrm{e}^{-\hbar\omega_\kappa/k_B T}) \tag{4.7}$$

where $đ\kappa$ is an integration over wave vectors and a sum over (transverse) polarizations. The free energy (density per spatial volume) is therefore given by

$$\frac{F}{V} = k_B T \frac{2 \times 4\pi}{(2\pi c)^3} \int_0^\infty \mathrm{d}\omega \, \omega^2 \log(1 - \mathrm{e}^{-\hbar\omega/k_B T}) \tag{4.8}$$

Show that this integral is convergent and proportional to $T\lambda_T^{-3}$ with the 'thermal wavelength' $\lambda_T = \hbar c/k_B T$.

Problem 4.2 – Thermal radiation, qualitatively (8 points + 5 bonus points)

(1) Compute the average photon number at room temperature for visible light and for the frequency band used by your mobile phone.

(2) Calculate the thermal wavelength λ_T at the photosphere temperature of the Sun. In which frequency band does this fall? Check the literature or the web on the 'Wien wavelength' and the peak of the Planck spectrum. Compare spectra vs wavelength and vs frequency.

(3) Convince yourself that the energy the Earth receives from the Sun by radiation (per day, per area) is exactly balanced by the (thermal) radiation of the Earth. What is not exactly balanced is the entropy $\Delta S = \Delta Q/T$ (Clausius). Look up the key word 'solar constant' and estimate the amount of entropy the Earth receives from the Sun (per day, per area). Normalized to $k_B \log 2$, this is the amount of information that is lost by the Earth.

(4) When the Universe was about three minutes old, it was in thermal equilibrium at a temperature T_e comparable to the rest mass of the electron. Estimate this temperature and the corresponding energy density. Which particles existed in significant numbers in this era?

(5) Consider protons and neutrons at the temperature of (3) and speculate why the ratio $\exp[(m_p - m_n)c^2/k_B T_e]$ gives an idea of the ratio of proton to neutron densities. How big is the number?

(6) Electrons, protons, and neutrinos are fermions. Each mode (momentum state incl spin label) can only be occupied by zero or one particle. Think about the partition function for massless neutrinos and argue that their density of free energy is given by

$$\frac{F_\nu}{V} = -k_B T \frac{2 \times 4\pi}{(2\pi c)^3} \int_0^\infty \mathrm{d}\omega \, \omega^2 \log(1 + \mathrm{e}^{-\hbar\omega/k_B T}) \tag{4.9}$$