Einführung in die Quantenoptik I

Wintersemester 2014/15
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Problem 5.1 – Coherent states (10 points)

In quantum optics, coherent (or Glauber) states are very popular. They are used to represent as closely as possible a field mode in a 'classical state'.

(i) Show that for any complex number $\alpha \in \mathbb{C}$, the state

$$|\alpha\rangle = \frac{1}{\mathcal{N}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{5.1}$$

is an eigenstate of the annihilation operator a:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{5.2}$$

and calculate the normalization factor $\mathcal{N} = \mathrm{e}^{|\alpha|^2/2}$. [5 bonus points:] Show that there are no eigenstates of the creation operator $a^\dagger$.

(ii) The coherent state has the property that expectation values of any quadrature are nonzero:

$$\langle\alpha|X_\theta|\alpha\rangle = \langle\alpha|\frac{a\,\mathrm{e}^{\mathrm{i}\theta} + a^\dagger\,\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{2}}|\alpha\rangle = \frac{\alpha\,\mathrm{e}^{\mathrm{i}\theta} + \alpha^*\,\mathrm{e}^{-\mathrm{i}\theta}}{\sqrt{2}} \tag{5.3}$$

(iii) Show that coherent states are not stationary, but evolve under the free field Hamiltonian according to

$$U(t)|\alpha\rangle = \mathrm{e}^{\mathrm{i}\phi(t)}|\alpha\,\mathrm{e}^{-\mathrm{i}\omega t}\rangle \tag{5.4}$$

where ω is the eigenfrequency of the mode. Calculate the phase $\phi(t)$ and find out under which conditions it remains unobservable. Comment on the sentence: 'Coherent states evolve according to the classical equations of motion of an oscillator.'

(iv) Coherent states are not orthogonal (why?) and 'overcomplete'. Show that for $\alpha, \beta \in \mathbb{C}$,

$$\text{not orthogonal:} \quad \langle\alpha|\beta\rangle = \mathrm{e}^{\mathrm{i}\phi(\alpha,\beta)} \exp(-\tfrac{1}{2}|\alpha-\beta|^2) \tag{5.5}$$

where the phase $\phi(\alpha, \beta)$ is antisymmetric under the exchange $\alpha \leftrightarrow \beta$ ('as it must be' – why?). Finally, using an integral $\mathrm{d}^2\alpha = \mathrm{d}x\,\mathrm{d}y$ (with $\alpha = x + \mathrm{i}y$) in the complex plane, show that

$$\text{overcomplete:} \quad \int \mathrm{d}^2\alpha\,|\alpha\rangle\langle\alpha| = \pi \sum_{n=0}^{\infty} |n\rangle\langle n| = \pi\mathbb{1} \tag{5.6}$$

The states are called overcomplete because of the factor $\pi > 1$.

Hint. Polar coordinates in the complex plane $\alpha = r\,\mathrm{e}^{\mathrm{i}\varphi}$ are very useful here.

Problem 5.2 – Quasi-probabilities (10 points)

The phase-space representation provides very useful pictures for quantum states and their time evolution. Coherent states play an important role here.

(i) P-function (Glauber-Sudarshan).

The P-function provides an expansion of a density operator in terms of projectors onto coherent states:

$$\rho = \int \mathrm{d}^2\alpha\, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha| \tag{5.7}$$

In the following, we write for brevity $P(\alpha)$.

This representation is not the most general one because it may be necessary to include 'skew projectors' of the type $|\alpha\rangle\langle\beta|$. This leads to the 'positive P-distribution'. For some tips and tricks, see *Numerical representation of quantum states in the positive-P and Wigner representations* by M. K. Olsen and A. S. Bradley, *Opt. Commun.* **282** (2009) 3924.

Show that a normally ordered operator product can be averaged 'in the intuitive way' with the P-function

$$\langle a^{\dagger m} a^n \rangle_\rho = \mathrm{tr}(a^{\dagger m} a^n\, \rho) = \int \mathrm{d}^2\alpha\, \alpha^{*m} \alpha^n P(\alpha) \tag{5.8}$$

For a coherent state, $\rho = |\beta\rangle\langle\beta|$, argue that $P(\alpha) = \delta^{(2)}(\alpha - \beta)$. (This is trivial, right?)

(ii) Q-function (Husimi).

Here, we deal with anti-normally ordered averages that are 'intuitive':

$$\langle a^m a^{\dagger n} \rangle_\rho = \int \mathrm{d}^2\alpha\, \alpha^m \alpha^{*n} Q(\alpha) \tag{5.9}$$

Show that this formula is consistent with the definition

$$Q(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle \tag{5.10}$$

using the over-completeness relation (5.6). Show that $1/\pi \geq Q(\alpha) \geq 0$ and calculate the Q-function of the vacuum state:

$$Q_{\mathrm{vac}}(\alpha) = \frac{\mathrm{e}^{-|\alpha|^2}}{\pi} \tag{5.11}$$

Show that P- and Q-functions are related by

$$Q(\alpha) = \int \frac{\mathrm{d}^2\beta}{\pi}\, \mathrm{e}^{-|\alpha-\beta|^2} P(\beta) \tag{5.12}$$

and check that this is a 'Gaussian convolution'. Intuitive interpretation?