Einführung in die Quantenoptik I

Wintersemester 2014/15
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Übungsaufgaben Blatt 6 Ausgabe: 16. Dezember 2014
Abgabe: 13. Januar 2015

Problem 6.1 – Non-resonant classical source (6 points)

In the lecture, you have heard about the statement 'A classical source generates a coherent state.' (1) Explain why the following Hamiltonian describes a quantized mode coupled to a classical source

$$H(t) = \hbar\omega_c a^\dagger a - f(t)(a + a^\dagger) \tag{6.1}$$

(2) Write down the Heisenberg equations of motion for the annihilation operator $a(t)$ and show that a possible solution is given by

$$a(t) = a(0)\,\mathrm{e}^{-\mathrm{i}\omega_c t} + g \int_0^t \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}\omega_c(t-t')} f(t') \tag{6.2}$$

where you still have to fix the complex constant g. Evaluate $\langle a(t)\rangle$ with the vacuum state as initial condition (time 0) and justify why the result is consistent with the generation of a coherent state at time t. (3) Specialize to a monochromatic source (frequency ω: $f(t) = f\,\mathrm{e}^{-\mathrm{i}\omega t} + \text{c.c.}$) and observe that $a(t)$ contains 'resonant' and 'non-resonant' terms proportional to $1/(\omega - \omega_c)$ and $1/(\omega + \omega_c)$, respectively. Estimate the ratio of these terms for a source in the visible region, with a detuning of $1\,\mathrm{nm}$ in wavelength.

Problem 6.2 – Interaction picture, rotating frame (8 points)

Consider a driven Hamiltonian in the near-resonant approximation,

$$H(t) = \hbar\omega_c a^\dagger a - (f^* a\,\mathrm{e}^{\mathrm{i}\omega t} + f a^\dagger\,\mathrm{e}^{-\mathrm{i}\omega t}) \tag{6.3}$$

and make the following *Ansatz* to solve the time-dependent Schrödinger equation

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\varpi t a^\dagger a}|\tilde{\psi}(t)\rangle \tag{6.4}$$

(1) Show that $|\tilde{\psi}(t)\rangle$ solves a Schrödinger equation with the following 'effective Hamiltonian'

$$\tilde{H}(t) = \hbar(\omega_c - \varpi)a^\dagger a - (f^* a\,\mathrm{e}^{\mathrm{i}(\omega-\varpi)t} + f a^\dagger\,\mathrm{e}^{-\mathrm{i}(\omega-\varpi)t}) \tag{6.5}$$

(2) Discuss the advantages and disadvantages of the choices $\varpi = \omega$ and $\varpi = \omega_c$. (3) Make the choice $\varpi = \omega$ and show that the ('effective') time evolution operator is given by

$$\begin{aligned} \tilde{U}(t) &= \mathrm{e}^{-\mathrm{i}\delta E t/\hbar} \exp[\mathrm{i}\Delta t (a^\dagger + \alpha^*)(a + \alpha)] \\ \Delta &= \omega - \omega_c \\ \alpha, \delta E &= ? \end{aligned} \tag{6.6}$$

(4) Compare δE to the level shift of the vacuum state $|0\rangle$ in second-order perturbation theory, calculated from $\tilde{H}$ for $\varpi = \omega$:

$$\delta E_0 = \frac{|\langle 1|(f^* a + f a^\dagger)|0\rangle|^2}{-\hbar\Delta} \tag{6.7}$$

Problem 6.3 – Campbell, Baker, Hausdorff, next please! (6 points)

Use the Campbell-Baker-Hausdorff identity for two operators A, B

$$\mathrm{e}^{A+B}\mathrm{e}^{\frac{1}{2}[A,B]} = \mathrm{e}^A \mathrm{e}^B, \qquad \text{provided } [A,B] \text{ commutes with } A \text{ and } B \tag{6.8}$$

(and suitable other tricks) to prove the following formulas for the displacement operators

$$D(\alpha) := \exp(\alpha a^\dagger - \alpha^* a) = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \exp(\alpha a^\dagger) \exp(-\alpha^* a) \tag{6.9}$$

$$D(\alpha)|0\rangle = |\alpha\rangle \tag{6.10}$$

$$D(\alpha)D(\beta) = \mathrm{e}^{\alpha^*\beta - \beta^*\alpha} D(\alpha + \beta) \tag{6.11}$$

$$D^\dagger(\alpha) = D(-\alpha) = D^{-1}(\alpha) \tag{6.12}$$

$$\exp(\mathrm{i}\theta a^\dagger a) D(\alpha) \exp(-\mathrm{i}\theta a^\dagger a) = D(\alpha\,\mathrm{e}^{\mathrm{i}\theta}) \quad \text{or} \quad D(\alpha\,\mathrm{e}^{-\mathrm{i}\theta}) \tag{6.13}$$

$$D^\dagger(\alpha) \exp(\mathrm{i}\theta a^\dagger a) D(\alpha) = \exp[\mathrm{i}\theta (a^\dagger + \alpha^*)(a + \alpha)] \tag{6.14}$$